

PERPENDICULAR INDISCERNIBLE SEQUENCES IN REAL CLOSED FIELDS

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§ 0. INTRODUCTION

In this paper we investigate the behaviour of concepts from dependent theories when applied to real closed fields. Our main focus will be in the concept of perpendicular indiscernible sequences, a concept first introduced in [Sh:715, §4].

A (first-order) theory is dependent when it's monster model doesn't contain a sequence of finite sequences $\{\bar{a}_i\}_{i \in \omega}$ and a formula $\varphi(x, y)$ with $\ell(y) = \ell(\bar{a}_i)$ for $i < \omega$ such that for every finite set $S \subseteq \omega$ and a "truth requirement function" $\eta : S \rightarrow \{T, F\}$ there exists some x_η with $\models \bigwedge_{i \in S} \varphi[x_\eta, \bar{a}_i]^{\eta(i)}$. This concept was

introduced by Shelah, see [Sh:715]. This concept is related to stability of models, and one can show that every stable theory is also dependent.

Perpendicularity in dependent models (models of a dependent theory) is a binary relation between infinite indiscernible sequences. Two such sequences $\bar{\mathbf{a}}^1 = \langle a_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle a_t^2 : t \in I^2 \rangle$ are said to be perpendicular if for every formula $\varphi(x, y)$ with $\ell(x) = \ell(a_t^1), \ell(y) = \ell(a_t^2)$ there exists a truth value \mathbf{t} such that for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[a_t, a_s]^{\mathbf{t}}$ and for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[a_t, a_s]^{\mathbf{t}}$, see Definition 2.8. This concept is suggested in [Sh:715, 4] as a substitute for orthogonal sequences in stable theories.

A real closed field is an ordered field which exhibits the Intermediate Value Theorem for polynomials, i.e. for every polynomial $p(x)$ and elements a, b such that $p(a) > 0, p(b) < 0$ there exists some c in the interval (a, b) such that $p(c) = 0$. Tarski proved that this theory has quantifier elimination, and so it is easily concluded that the theory is dependent. See "Preliminaries".

In this paper we investigate perpendicularity of indiscernible sequences in real closed fields. In §2 we properly define perpendicularity and supply equivalent definitions. We also characterize perpendicular sequences based on results from [Sh:715]. In §3 we will review the subject of Dedekind cuts in real closed Dedekind fields, emphasizing the notion of dependent cuts from the model theoretic perspective. In §4 we connect the concepts of perpendicularity of sequences and dependency of cuts in real closed fields, and we show that under certain condition cuts are dependent iff the sequences inducing them are not dependent (Claim 4.10 and Theorem 4.5). In §5 we define strong perpendicularity between indiscernible sequences in dependent models, and we prove that in real closed fields no two indiscernible sequences are strongly perpendicular (Theorem 5.17).

We presume the reader is well familiar with model theory and abstract model theory and fairly familiar with real closed fields. A good introduction to model theory can be found in [Hod97]. A good introduction to real closed fields can be found in [DMP06]. We also recommend the reader to familiarize himself or herself with the definitions and results which appeared in [Sh:715] and [Sh:783], although we will state and prove every definition or claim we use from there.

§ 1. PRELIMINARIES

In this section we introduce the basic concepts on which the rest of the work depends and several basic known results. The section is divided into two parts. In the first part we define the concept of “dependent theories” and dependent formulas. In the second part we cite Tarski’s theorem of quantifier elimination in the theory of real closed field, and several immediate conclusions which will be helpful in the next items. The reader is invited to skip this section and use it as a reference.

§ 1(A). A Word About Notation.

We shall work mainly with the theory of real closed fields. However, when the theory under discussion is known we will denote by \mathcal{C} the “monster model”. As usual, φ^0 means $\neg\varphi$ and φ^1 means φ .

§ 1(B). Dependent Theories.

Definition 1.1. Let T be a complete theory. T is said to be independent if for some formula $\varphi(\bar{x}, \bar{y})$, possibly with parameters, $\varphi(\bar{x}, \bar{y})$ is independent, which means that for every $n < \omega$:

$$T \vdash \exists \bar{y}_0 \exists \bar{y}_1 \dots \exists \bar{y}_{n-1} \bigwedge_{\eta \in \{0,1\}^n} \exists \bar{x} \left(\bigwedge_{k < n} \varphi(\bar{x}, \bar{y}_k)^{\eta(k)} \right).$$

T is said to be dependent if it’s not independent. A model is called dependent if its theory is.

Example 1.2. 1) Let $\langle V, E \rangle$ be a ***-random graph**. Then $\text{Th}(\langle V, E \rangle)$ is independent.

2) The theory of dense linear order is dependent.

Proof. 1) A *-random graph $\langle V, E \rangle$ is an infinite graph $|V| \geq \aleph_0$ where every finite graph can be embedded into it. Here $\varphi(x, y) = xEy$ is independent.

2) The theory of dense linear order can be interpreted in the theory of real closed fields. Now use the next Lemma and Claim 1.7. \square

Lemma 1.3. *Let M be some dependent model and let N be a model interpreted in M . Then N is dependent.*

Proof. Denote by $\rho : N \rightarrow M^k, \rho : \tau(N) \rightarrow \mathbb{L}(M)$ the interpretation function. So ρ induces a function $\rho' : \mathbb{L}(N) \rightarrow \mathbb{L}(M)$ (and we use the notation $\varphi_\rho = \rho'(\varphi)$) such that $N \models \psi[\bar{a}] \Leftrightarrow M \models \psi_\rho[\rho(\bar{a})]$. Now assume by contradiction that the claim is *false*. So for some formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(N)$ we have that ρ is independent. We claim that φ_ρ is independent in M . Let $n < \omega$. By assumption there exists $\bar{y}_0, \dots, \bar{y}_{n-1} \in {}^{\ell g(\bar{y})}N$ such that

$$N \models \bigwedge_{\eta \in \{0,1\}^n} \exists \bar{x} \left(\bigwedge_{k < n} \varphi(\bar{x}, \bar{y}_k)^{\eta(k)} \right).$$

So we have

$$M \models \bigwedge_{\eta \in \{0,1\}^n} \exists \bar{x}' \left(\bigwedge_{k < n} \varphi_\rho(\bar{x}', \rho(\bar{y}_k))^{\eta(k)} \right).$$

Where $\ell g(\bar{x}') \cdot k$. This shows that φ_ρ is independent in M . Contradiction. \square

Definition 1.4. Let I be some linearly ordered set and let $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ be some indiscernible sequence with $\ell g(\bar{a}_t) = m$ for every $t \in I$. A formula $\varphi(\bar{x}, \bar{y})$ possibly with parameters, $\ell g(\bar{y}) = m$ is said to be dependent relative to $\bar{\mathbf{a}}$, if for every $\bar{b} \in {}^{\ell g(\bar{x})}\mathcal{C}$ the set $\{t \in I : \models \varphi[\bar{b}, \bar{a}_t]\}$ is a finite union of convex subsets of I . We define $\text{dpfor}(\bar{\mathbf{a}}) \subseteq \mathbb{L}(T)$ to be the set of such formulas.

Claim 1.5. Let T be a dependent theory, and $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ some indiscernible sequence. Then $\text{dpfor}(\bar{\mathbf{a}}) = \{\varphi(\bar{x}, \bar{y}) \subseteq \mathbb{L}(T) : \ell g(\bar{y}) = \ell g(\bar{a}_t)\}$. Furthermore, for every $\varphi(\bar{x}, \bar{y})$ there exists $k_\varphi \in \omega$ such that for every indiscernible sequence $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ and every $\bar{b} \in {}^{\ell g(\bar{x})}\mathcal{C}$ we have that $\{t \in I : \models \varphi[\bar{b}, \bar{a}_t]\}$ is a union of no more than k_φ convex subsets of I .

Proof. T is dependent and so by definition there exists $n < \omega$ such that:

$$T \models \neg \exists \bar{y}_0 \exists \bar{y}_1 \dots \exists \bar{y}_{n-1} \bigwedge_{\eta \in \{0,1\}^n} \exists \bar{x} \left(\bigwedge_{k < n} \varphi(\bar{x}, \bar{y}_k)^{\eta(k)} \right).$$

Now assume toward contradiction $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence and $\bar{b} \in {}^{\ell g(\bar{x})}\mathcal{C}$ such that $\langle \varphi[\bar{b}, \bar{a}_t] : t \in I \rangle$ change signs $2n - 1$ times. Without loss of generality there exist $t_0 <_I t_1 <_I \dots <_I t_{2n-1}$ such that $\models \varphi[\bar{b}, \bar{a}_{t_k}]$ if k odd. Now according to the inset equation above there exists some $\eta \in \{0,1\}^n$ such that $\models \forall \bar{x} (\neg \bigwedge_{k < n} \varphi(\bar{x}, \bar{a}_{t_k})^{\eta(k)})$, using indiscernibility $\models \forall \bar{x} (\neg \bigwedge_{k < n} \varphi(\bar{x}, \bar{a}_{t_{2k+\eta(k)}})^{\eta(k)})$, in contradiction to $\models \bigwedge_{k < n} \varphi[\bar{b}, \bar{a}_{t_{2k+\eta(k)}}]^{\eta(k)}$, so $k_\varphi = 2n - 1$ satisfies the requirements. \square

§ 1(C). Real Closed Field.

We assume some background in the definitions of real closed fields. Throughout this paper, \mathcal{F} will denote a model of a real closed field.

We shall use the next theorem by Tarski quite often. We bring it here without proof.

Theorem 1.6. (*Quantifier elimination for the theory of real closed fields*)

The theory of real closed fields has quantifier elimination in the language $\langle +, \cdot, 0, 1, < \rangle$.

Proof. See [DMP06, Ch.1]. \square

Claim 1.7. *The theory of $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$, i.e. the theory of real closed fields is dependent.*

Proof. We work in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$. By the previous theorem, we reduce to the following case.

Assume for a contradiction that for some polynomial p in $s + m$ variables we have $p(\bar{x}, \bar{y}, \bar{c}) > 0$ is independent with $\ell g(\bar{x}) = s, \ell g(\bar{y}) = m$. Rewrite $p(\bar{x}, \bar{y}) = \sum_{i=0}^{k-1} m_i(\bar{x}, \bar{y}, \bar{c})$ where the m_i 's are monomials. By definition of independent formula there exist $\bar{b}_0, \dots, \bar{b}_k$ such that for any function η with domain $\{0, \dots, k\}$ and range $\{0, 1\}$,

$$\models \exists \bar{x} \left(\bigwedge_{i \leq k} p(\bar{x}, \bar{b}_i, \bar{c}) \cdot (-1)^{\eta(i)} > 0 \right).$$

Denote by d_i^j the coefficient of \bar{x} in the monomial $m_i(\bar{x}, \bar{b}_j, \bar{c})$ and $\bar{d}^j = (d_0^j, d_1^j, \dots, d_{k-1}^j)$. Now by counting dimensions we know that $\{\bar{d}^j : j \in \{0, \dots, k\}\}$ is linearly dependent over \mathbb{R} , namely $\sum_{j=0}^k a_j \bar{d}^j = 0$, with $(a_0, \dots, a_k) \neq \bar{0}$. So we have $\sum_{j=0}^k a_j \cdot p(\bar{x}, \bar{b}_j, \bar{c}) = 0$. Choose η as above so that $\eta(j) = 0$ iff $a_j > 0$. Now we assumed that for each such η there is an \bar{x}_* such that

$$\begin{aligned} \bigwedge_{j \leq k} p(\bar{x}_*, \bar{b}_j, \bar{c}) \cdot (-1)^{\eta(j)} &> 0 \text{ thus} \\ \bigwedge_{j \leq k} p(\bar{x}_*, \bar{b}_j, \bar{c}) \cdot a_j &\geq 0 \text{ with some polynomial strictly greater than } 0 \\ \sum_{j \leq k} a_j \cdot p(\bar{x}_*, \bar{b}_j, \bar{c}) &> 0. \end{aligned}$$

This contradicts the definition of (a_0, \dots, a_k) so we finish. \square

We shall use the following corollaries of quantifier elimination later on:

Corollary 1.8. *Let \mathcal{F} be some real closed field and $\phi(x, \bar{a})$ a formula with parameters from \mathcal{F} . Then the set $\{x \in \mathcal{F} : \models \phi[x, \bar{a}]\}$ is a finite union of intervals of \mathcal{F} . Furthermore, the number of intervals is bounded uniformly independent of \bar{a} .*

Corollary 1.9. *Let $f(x) : \mathcal{F} \rightarrow \mathcal{F}$ be some function definable over \mathcal{F} . Then f is piecewise monotonic, with each piece either constant or strictly monotonic. In other words, there exist $a_0 < \dots < a_{n+1}$ with $a_0 = -\infty, a_{n+1} = \infty$ such that f is constant or strictly monotonic on (a_i, a_{i+1}) for $i \in \{0, \dots, n\}$.*

Corollary 1.10. *Let \mathcal{F} be some real closed field and $a, b \in \mathcal{F}$ realize the same type over some $A \in \mathcal{F}$. Then for all $d \in (a, b)_{\mathcal{F}}$ we have that d realizes the same type over A as a, b .*

Proof. Assume otherwise and let $\varphi(x)$ witness: $\models (\neg \varphi[d]) \wedge \varphi[a]$. By the previous corollaries φ divides the field into finitely many intervals, so define $\psi_n(x) = \text{"}\varphi(x) \text{ and } x \text{ is in the } n\text{-th interval realizing } \varphi(x)\text{"}$. It is easy to see that the ψ_n 's use only parameters from A so for some n we have that $\models \psi_n[a]$. So by assumption $\models \psi_n[b]$ this contradicts the existence of $d \in (a, b)_{\mathcal{F}}$. \square

§ 2. PERPENDICULARITY IN DEPENDENT THEORIES

In this item we define and explore the notion of perpendicular indiscernible sequence in dependent theories. This notion is somewhat parallel to the notion of orthogonal sequences in stable theories (see [Sh:c, Ch.V]). We also introduce a technique for constructing indiscernible sequences based on ultra-filters or other indiscernible sequences. This technique is very important to the understanding of the rest of the work.

From here on we will denote by T a dependent theory.

Definition 2.1. 1) Let $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ be an endless indiscernible sequence (i.e. I has no last element). For every $A \subset \mathcal{C}$ and $\Delta \subseteq \mathbb{L}(T)$ we define the Δ -average of $\bar{\mathbf{a}}$ over A or $\text{Av}_\Delta(A, \bar{\mathbf{a}})$ to be the type:

$$\text{Av}_\Delta(A, \bar{\mathbf{a}}) = \{\varphi(\bar{x}, \bar{b})^t : \varphi(\bar{x}, \bar{y}) \in \Delta, \bar{b} \in {}^{\ell g(\bar{y})}A, \ell g(\bar{x}) = \ell g(\bar{a}_t), \\ \text{for every large enough } t \in I \text{ we have } \models \varphi[\bar{a}_t, \bar{b}]^t\}.$$

2) Let D be some ultrafilter on some ${}^m C \subset \mathcal{C}$. For every $A \subset \mathcal{C}$ we define the Δ -average of D over A or $\text{Av}_\Delta(A, D)$ to be the type:

$$\text{Av}_\Delta(A, D) = \{\varphi(\bar{x}, \bar{b})^t : \varphi(\bar{x}, \bar{y}) \in \Delta, \bar{b} \in {}^{\ell g(\bar{y})}A, \ell g(\bar{x}) = m, \\ \{\bar{c} \in {}^m \mathcal{C} : \models \varphi[\bar{c}, \bar{b}]^t\} \in D.\}$$

Omitting the Δ means for $\Delta = \mathbb{L}(T)$.

In Clause 2 of the above definition we think of ${}^m \mathcal{C}$ as a “test set” and we say $\varphi(\bar{x}, \bar{b}) \in \text{Av}_\Delta(A, D)$ if the “majority” of ${}^m C$, according to D , exhibit $\models \varphi[\bar{x}, \bar{b}]$. Since D is closed under finite disjunctions, $\text{Av}_\Delta(A, D)$ is finitely realized in ${}^m C$.

Remark 2.2. Note that in Definition 2.1 when writing $\text{Av}_\Delta(A, \bar{\mathbf{a}})$ the parameter set A comes before the object $\bar{\mathbf{a}}$ in question, in keeping with usage in published articles of the second author.

Conclusion 2.3. *If T is dependent and I is endless, then $\text{Av}(A, \bar{\mathbf{a}})$ is a complete type over A .*

Proof. Follows from Claim 1.5. □

Definition 2.4. Let T be dependent, $\bar{\mathbf{a}}$ a Δ -indiscernible sequence in \mathcal{C} and $A \subset \mathcal{C}$ with $\bar{\mathbf{a}} \subseteq A$. We say that $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is Δ -based on $\bar{\mathbf{a}}$ over A if for every $t \in I$ we have that \bar{b}_t realizes $\text{Av}_\Delta(A \cup \{\bar{b}_s : s <_I t\}, \bar{\mathbf{a}})$. Omitting the Δ means for $\Delta = \mathbb{L}(T)$.

Example 2.5. Let T be the theory of dense linear order with a model \mathbb{Q} and let $\bar{\mathbf{a}} = \langle 10 - \frac{1}{n} : n \in \mathbb{N} \rangle$. We will find a sequence based on $\bar{\mathbf{a}}$ over \mathbb{Q} in some real closed field extending \mathbb{Q} . To that end, we will need to find an element which realize $\{0 < x < p : 0 < p \in \mathbb{Q}\}$, denoted δ . Now the reader can check that the sequence $\langle 10 - n\delta : n \in \mathbb{N} \rangle$ is as needed.

Claim 2.6. 1) *If $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are Δ -based on $\bar{\mathbf{a}} = \langle \bar{a}_s : s \in J \rangle$ over some A with the same order type I , then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ have the same Δ -type over A .*

2) *If $\bar{\mathbf{b}}$ is Δ -based on $\bar{\mathbf{a}}$ over some A then $\bar{\mathbf{b}}$ is Δ -indiscernible over A .*

Remark 2.7. In the Definition 2.4 of Δ -based we require $\bar{\mathbf{a}} \subseteq A$.

Proof. 1) Assume toward contradiction that there exists $\varphi \in \Delta, t_1 <_I \dots <_I t_{k_\varphi}, \bar{d}_\varphi \subseteq A$ with $\models \varphi[\bar{b}_{t_1}^1, \dots, \bar{b}_{t_{k_\varphi}}^1, \bar{d}_\varphi]$ but $\models \neg \varphi[\bar{b}_{t_1}^2, \dots, \bar{b}_{t_{k_\varphi}}^2, \bar{d}_\varphi]$. Take such φ with minimal k_φ . By minimality of k_φ we have that $\bar{b}_{t_1}^1, \dots, \bar{b}_{t_{k_\varphi-1}}^1, \bar{b}_{t_1}^2 \dots \bar{b}_{t_{k_\varphi-1}}^2$ realize the same Δ -type over A . By definition of average we know that for every large enough $s \in J$ we have that $\models \varphi[\bar{b}_{t_1}^1, \dots, \bar{b}_{t_{k_\varphi-1}}^1, \bar{a}_s, \bar{d}_\varphi]$, hence also for every large enough $s \in J$ we have $\models \varphi[\bar{b}_{t_1}^2, \dots, \bar{b}_{t_{k_\varphi-1}}^2, \bar{a}_s, \bar{d}_\varphi]$. Now by the definition of average we get $\models \varphi[\bar{b}_{t_1}^2, \dots, \bar{b}_{t_{k_\varphi-1}}^2, \bar{b}_{t_k}^2, \bar{d}_\varphi]$, contradicting the assumptions.

2) Follows from Clause 1 noticing that every finite subsequence of $\bar{\mathbf{b}}$ is actually a sequence based on $\bar{\mathbf{a}}$ over A . \square

Definition 2.8. 1) We say that two sequences $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are Δ -mutually indiscernible over A if for $\ell = 1, 2$ we have that $\bar{\mathbf{b}}^\ell$ is Δ -indiscernible over $\bar{\mathbf{b}}^{3-\ell} \cup A$.

2) Omitting A means for $A = \phi$. Omitting Δ means $\Delta = \mathbb{L}(T)$.

3) We say that two endless Δ -indiscernible sequences $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are Δ -perpendicular iff for every $A \subset \mathcal{C}$, for some $\langle \bar{b}_n^\ell : \ell \in \{1, 2\}, n < \omega \rangle$ where \bar{b}_n^ℓ realizes $\text{Av}_\Delta(A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\}, \bar{\mathbf{a}}^\ell)$ for $\ell \in \{1, 2\}, n < \omega$ we have that $\langle \bar{b}_n^1 : n < \omega \rangle, \langle \bar{b}_n^2 : n < \omega \rangle$ are Δ -mutually indiscernible over $A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2$. Omitting the Δ means for $\Delta = \mathbb{L}(T)$.

Example 2.9. In the theory of dense linear order, two strictly increasing indiscernible sequences of elements are perpendicular iff they induce different cuts (see below).

Claim 2.10. In Definition 15, Clause 3 replacing “for some $\langle \bar{b}_n^\ell : \ell \in \{1, 2\}, n < \omega \rangle$ ” with “for every $\langle \bar{b}_n^\ell : \ell \in \{1, 2\}, n < \omega \rangle$ ” is equivalent.

Proof. We will deal with the non-trivial direction only. It is enough to prove that for any two pairs of sequences $\langle \bar{b}_n^{\ell, i} : \ell \in \{1, 2\}, n < \omega \rangle$, constructed as in Definition 2.8(3), i.e. with $\bar{b}_n^{\ell, i}$ realizing $\text{Av}_\Delta(A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\}, \bar{\mathbf{a}}^\ell)$, we have that $\langle \bar{b}_n^{\ell, i} : \ell \in \{1, 2\}, n < \omega \rangle$ realize the same type over $A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2$ for $i \in \{1, 2\}$. Define a new order \prec on these sequences: $\bar{b}_n^{\ell, i} \prec \bar{b}_m^k$ iff $n < m$ or $(m = n \wedge \ell < k)$. From here the proof is similar to the proof of Claim 2.6. \square

Claim 2.11. Being perpendicular and being Δ -perpendicular are both symmetric notions.

Proof. Assume $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are Δ -perpendicular and we will prove that $\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^1$ are Δ -perpendicular. So let $\langle \bar{b}_n^\ell : \ell \in \{1, 2\}, n < \omega \rangle$ be such that \bar{b}_n^ℓ realizes

$$\text{Av}_\Delta(A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\}, \bar{\mathbf{a}}^\ell)$$

for $\ell \in \{1, 2\}, n < \omega$. By Claim 17 $\langle \bar{b}_n^\ell : n < \omega \rangle, \langle \bar{b}_n^2 : n < \omega \rangle$ are Δ -mutually indiscernible. Define $\bar{c}_n^1 = \bar{b}_{n+1}^1$ for $n < \omega$ and $\bar{c}_n^2 = \bar{b}_n^2$ for $n < \omega$. Now easily $\langle \bar{c}_n^2 : n < \omega \rangle, \langle \bar{c}_n^1 : n < \omega \rangle$ are witnesses that $\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^1$ are Δ -perpendicular. \square

Claim 2.12. Assume $\bar{\mathbf{a}}^1 = \langle \bar{a}_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle \bar{a}_s^2 : s \in I^2 \rangle$ are Δ -indiscernible and $J^\ell \subseteq I^\ell$ is unbounded in I^ℓ for $\ell \in \{1, 2\}$. So $\bar{\mathbf{c}}^1 = \langle \bar{a}_t^1 : t \in J^1 \rangle, \bar{\mathbf{c}}^2 = \langle \bar{a}_s^2 : s \in J^2 \rangle$ are Δ -perpendicular iff $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are Δ -perpendicular.

Proof. This follows from the following fact: if $\langle \bar{a}_t : t \in I \rangle$ is Δ -indiscernible and $J \subseteq I$ is unbounded, then for every $A \subset \mathcal{C}$ we have $\text{Av}_\Delta(A, \langle \bar{a}_t : t \in I \rangle) = \text{Av}_\Delta(A, \langle \bar{a}_t : t \in J \rangle)$. \square

Claim 2.13. *Let $\langle a_t^1 : t \in I^1 \rangle, \langle \bar{a}_t^2 : t \in I^2 \rangle$ be two indiscernible sequences. Then for every formula $\varphi(\bar{x}, \bar{y})$ possibly with parameters there exists a truth value \mathbf{t} such that for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2]^{\mathbf{t}}$.*

Proof. Assume the contrary. Then by Claim 1.5 for any $n \in \omega$ there exists an increasing sequence $t_0 <_{I^1} \dots <_{I^1} t_{n-1}$ such that for any $j = 0, \dots, n-1$ for every large enough $s \in I^2$ we have that $\models \varphi[\bar{a}_{t_j}^1, \bar{a}_s^2]$ iff j odd. n is finite and therefore we can find $s \in I^2$ large enough for every $j < n$. Hence for every $n \in \omega$ there exists some $s \in I^2$ with $\{t \in I^1 : \models \varphi[\bar{a}_t^1, \bar{a}_s^2]\}$ contains no less than $\frac{n}{2}$ convex subsets of I^1 , contradicting the Claim 1.5. \square

Remark 2.14. While Claim 2.13 is symmetric, note that in Claim 2.15(4) we choose \mathbf{t} in advance which must cohere for both directions.

Claim 2.15. *Let $\bar{\mathbf{a}}^1 = \langle \bar{a}_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle \bar{a}_t^2 : t \in I^2 \rangle$ be two indiscernible sequences. Then the following are equivalent:*

- (1) $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular
- (2) for every $\langle \bar{b}_n^\ell : \ell = 1, 2; n \in \omega \rangle$ such that each \bar{b}_n^ℓ realizes $\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\}, \bar{\mathbf{a}}^\ell)$ we have that $\langle \bar{b}_n^1 : n \in \omega \rangle, \langle \bar{b}_n^2 : n \in \omega \rangle$ are mutually indiscernible
- (3) for every $A \subset \mathcal{C}$ if \bar{b}^1 realizes $\text{Av}(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A)$ and \bar{b}^2 realizes $\text{Av}(\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A \cup \bar{b}^1)$ then \bar{b}^1 also realizes $\text{Av}(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A \cup \bar{b}^2)$
- (4) for every formula $\varphi(\bar{x}, \bar{y})$ possibly with parameters there exists a truth value \mathbf{t} such that for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2]^{\mathbf{t}}$ and for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2]^{\mathbf{t}}$
- (5) for some $(|\bar{\mathbf{a}}^1| + |\bar{\mathbf{a}}^2|)^+$ -saturated model \mathcal{F} containing $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ for every formula $\varphi(\bar{x}, \bar{y})$ possibly with parameters from \mathcal{F} there exists a truth value \mathbf{t} such that for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2]^{\mathbf{t}}$ and for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2]^{\mathbf{t}}$.

Proof. 1) We will show (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (5), (5) \rightarrow (4) \rightarrow (3) \rightarrow (1), which suffices.

2) Follows from (1) by choosing $A = \emptyset$ in Definition 2.8(3) and Claim 2.10.

Now assume (2) and by contradiction assume that (3) fails for some A . Now let $\langle \bar{b}_n^\ell : \ell = 1, 2; n \in \omega \rangle$ be such that \bar{b}_n^ℓ realizes $\text{Av}(A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\})$ for $\ell = 1, 2, n \in \omega$. (3) fails, so there exists some formula $\varphi(\bar{x}, \bar{y})$ with parameters from $A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2$ such that $\models \varphi[\bar{b}_k^1, \bar{b}_\ell^2]$ iff $k > \ell$. Now denote $\psi(\bar{x}, \bar{y}, \bar{c}) = \varphi(\bar{x}, \bar{y})$ such that $\psi(\bar{x}, \bar{y}, \bar{z})$ is parameter free. We claim that $\psi(\bar{x}, \bar{y}, \bar{z})$ is independent. So for every $n \in \omega$ and $\eta \in {}^n 2$ we can choose $k_0 < \dots < k_{n-1}$ and $\ell_0 < \dots < \ell_{n-1}$ such that for every $j = 0, \dots, n-1$ we have $k_j < \ell_j$ iff $\eta(j) = 0$. Now $\models \bigwedge_{j < n} \psi(\bar{b}_{k_j}^1, \bar{b}_{\ell_j}^2, \bar{c})^{\eta(j)}$ and by mutual-indiscernibility $\models \exists \bar{z} \bigwedge_{j < n} \psi(\bar{b}_j^1, \bar{b}_j^2, \bar{z})^{\eta(j)}$ so the formula $\psi(\bar{x}, \bar{y}, \bar{z})$ is independent, contradiction.

To prove (4) from (3) let $\varphi(\bar{x}, \bar{y}, \bar{c})$ be some formula with $\bar{c} \subseteq A$. Now let \bar{b}^1 realize $\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A, \bar{\mathbf{a}}^1)$ and let \bar{b}^2 realize $\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A \cup \bar{b}^1, \bar{\mathbf{a}}^2)$. By Claim 2.13 there exist truth values $\mathbf{t}^1, \mathbf{t}^2$ such that for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]^{\mathbf{t}^1}$ and for every large enough $s \in I^2$ for every

large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]^{\mathbf{t}^2}$. We will prove that $\mathbf{t}^1 = \mathbf{t}^2$. Now for every large enough $s \in I^2$ we have that $\models \varphi[\bar{b}^1, \bar{a}_s^2, \bar{c}]^{\mathbf{t}^2}$. By definition of average $\models \varphi[\bar{b}^1, \bar{b}^2, \bar{c}]^{\mathbf{t}^2}$. Now by (3) we have that \bar{b}^1 realize $\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A \cup \bar{b}^2, \bar{\mathbf{a}}^1)$, hence for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{b}^2, \bar{c}]^{\mathbf{t}^2}$. So for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]^{\mathbf{t}^2}$. So $\mathbf{t}^1 = \mathbf{t}^2$ and we are done.

(3) Follows from (4) is symmetric.

(5) Follows from (4) trivially.

To prove (4) from (5) assume toward contradiction that (4) fails. So some formula $\varphi(\bar{x}, \bar{y}, \bar{d})$ witness the failure of (4). Now let $\bar{e} \subset \mathcal{F}$ realize $\text{tp}(\bar{d}, \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2)$. So $\varphi(\bar{x}, \bar{y}, \bar{e})$ witness the failure of (5). Contradiction and so (4) follows from (5).

To (1) from (3) it is enough to notice that by (3) $\langle \bar{b}_n^1 : n \in \omega \rangle$ constructed as in Definition 2.8(3) is actually a sequence based on $\bar{\mathbf{a}}$ over $A \cup \langle \bar{b}_n^2 : n \in \omega \rangle$ and vice-versa. Hence by Claim 2.6 and Remark 2.7 the two sequences are mutually indiscernible and we are done. \square

We now introduce a weaker version of perpendiculartiy to use in the next section.

Definition 2.16. Two indiscernible sequences $\bar{\mathbf{a}}^1 = \langle \bar{a}_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle \bar{a}_s^2 : s \in I^2 \rangle$ will be called (Δ, A) -perpendicular where Δ is some set of formulas of the form $\varphi(\bar{x}, \bar{y}, \bar{z})$ with $\ell g(\bar{x}) = \ell g(\bar{a}_t^1), \ell g(\bar{y}) = \ell g(\bar{a}_s^2)$ and $\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \subseteq A$ iff for every $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta, \bar{d} \subset A$ with $\ell g(\bar{d}) = \ell g(\bar{z})$ for some truth value \mathbf{t} we have that for every large enough $g \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{d}]^{\mathbf{t}}$ and for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{d}]^{\mathbf{t}}$. Omitting the Δ means for $\Delta = \mathbb{L}(T)$.

As a corollary of 2.15 we obtain:

Corollary 2.17. 1) If M is $(|\bar{\mathbf{a}}^1| + |\bar{\mathbf{a}}^2|)^+$ -saturated then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are (Δ, M) -perpendicular iff they are Δ -perpendicular.

2) If $B \subseteq A, \Delta_B \subseteq \Delta_A$ and $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are (Δ_A, A) -perpendicular then they are (Δ_B, B) -perpendicular.

Claim 2.18. Let $\bar{\mathbf{a}}^1 = \langle \bar{a}_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle \bar{a}_t^2 : t \in I^2 \rangle$ be two A -perpendicular sequences. Assume that for some $\bar{\mathbf{b}}^1 = \langle \bar{b}_n^1 : n \in \omega \rangle, \bar{\mathbf{b}}^2 = \langle \bar{b}_n^2 : n \in \omega \rangle$ we have that $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2 \subseteq A$ and for every $n \in \omega, \ell \in \{1, 2\}$ we have that \bar{b}_n^ℓ realizes

$$\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_m^k : k \in \{1, 2\}, m < n \text{ or } (m = n \wedge k < \ell)\}, \bar{\mathbf{a}}^\ell).$$

Then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular.

Proof. By claim 2.15(2) it is enough to show that $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible. So it is enough to show that $\bar{\mathbf{b}}^1$ is based on $\bar{\mathbf{a}}^1$ over $\bar{\mathbf{b}}^2$ (the proof of the other direction is symmetric). This will follow if we show that for every $n \in \omega, m > n$ we have that \bar{b}_n^1 realize

$$\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_k^1 : k < n\} \cup \{\bar{b}_k^2 : k < m\}, \bar{\mathbf{a}}^1).$$

We will prove this by induction on $m \in \omega$. So assume that \bar{b}_n^1 realize

$$\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_k^1 : k < n\} \cup \{\bar{b}_k^2 : k < m\}, \bar{\mathbf{a}}^1).$$

Now we know that \bar{b}_m^2 realize

$$\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_s^k : k \in \{1, 2\}, s < m \text{ or } (s = m \wedge k = 1)\}, \bar{\mathbf{a}}^2).$$

Let $\varphi(\bar{x}, \bar{b}_m^2, \bar{c}) \in \text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_k^1 : k < n\} \cup \{\bar{b}_k^2 : k < m + 1\}, \bar{\mathbf{a}}^1)$ with $\bar{c} \subseteq \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_k^1 : k < n\} \cup \{\bar{b}_k^2 : k < m\} \subseteq A$. So for every large enough $t \in I^1$ we have that $\models \varphi[\bar{a}_t^1, \bar{b}_m^2, \bar{c}]$. Hence for every large enough $s \in I^2$ we have that $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]$. By A -perpendicularity we have that for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]$. So for every large enough $s \in I^2$ we have $\models \varphi[\bar{b}_n^1, \bar{a}_s^2, \bar{c}]$. Hence $\models \varphi[\bar{b}_n^1, \bar{b}_m^2, \bar{c}]$. So \bar{b}_n^1 realize $\text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}_k^1 : k < n\} \cup \{\bar{b}_k^2 : k < m + 1\}, \bar{\mathbf{a}}^1)$ and we are done. \square

Claim 2.19. *Let $\bar{\mathbf{a}}^1 = \langle \bar{a}_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle \bar{a}_s^2 : s \in I^2 \rangle$ be two not (Δ, A) -perpendicular sequences for some $\phi \neq A \subseteq \mathcal{C}$. Then $\text{cf}(\bar{\mathbf{a}}^1) = \text{cf}(\bar{\mathbf{a}}^2)$.*

Proof. Denote $\lambda = \text{cf}(\bar{\mathbf{a}}^1)$. We will construct two subsequence $\langle \bar{a}_{t_\alpha}^1 : \alpha \in \lambda \rangle, \langle \bar{a}_{s_\beta}^2 : \beta \in \lambda \rangle$ of $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ respectively such that both are unbounded. This will prove that $\text{cf}(\bar{\mathbf{a}}^1) = \text{cf}(\bar{\mathbf{a}}^2)$. First take some unbounded subsequence $\langle t'_\alpha : \alpha \in \lambda \rangle$ of I^1 . We construct the sequences by induction on λ . By definition of perpendicularity we have that for some $\varphi(\bar{x}, \bar{y}, \bar{c}) \in \Delta$;

- (1) for every large enough $t \in I^1$ for every large enough $s \in I^2$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]$
- (2) for every large enough $s \in I^2$ for every large enough $t \in I^1$ we have $\models \varphi[\bar{a}_t^1, \bar{a}_s^2, \bar{c}]$.

Now choose t_0 large enough as in (1) and larger than t'_0 and choose s_0 large enough as in (1) for t_0 and also large enough as in (2). Now assume we constructed the sequences up to some α . Choose $t_{\alpha+1}$ large enough as in (2) for s_α and larger than $t'_{\alpha+1}$ and choose $s_{\alpha+1}$ large enough as in (1) for $t_{\alpha+1}$. For limit ordinals α choose $t_\alpha > \max_{i < \alpha} (t_i)$ and larger than t'_α (there exists such t_α since $\text{cf}(\bar{\mathbf{a}}^1) = \lambda > \alpha$) and choose s_α large enough as in (1) for t_α . The construction is thus completed.

Now $\langle \bar{a}_{t_\alpha}^1 : \alpha \in \lambda \rangle$ is unbounded since $t_\alpha > t'_\alpha$ for every $\alpha \in \lambda$ and $\langle t'_\alpha : \alpha \in \lambda \rangle$ is unbounded. Assume by contradiction that $\langle s_\alpha : \alpha \in \lambda \rangle$ is bounded in I^2 . Then for some $s \in I^2$ we have that s is large enough as in (1) for every $t \in I^1$. This contradicts (2). Then $\langle s_\alpha : \alpha \in \lambda \rangle$ is unbounded and so $\text{cf}(\bar{\mathbf{a}}^2) \leq \text{cf}(\bar{\mathbf{a}}^1)$ by symmetry we are done. \square

§ 3. CUTS IN REAL CLOSED FIELDS

In this section we explore cuts in real closed fields, more specifically from the model theoretic point of view. We define dependency of cuts and review several equivalent definitions and results. The last result in this section (Claim 3.14) states that two cuts in a real closed field \mathcal{F} are dependent if some polynomial $p(x, y) \neq 0$ with coefficients from \mathcal{F} has a sequence of roots $\langle (a_t, b_t) : t \in I \rangle$ with $\langle a_t : t \in I \rangle, \langle b_t : t \in I \rangle$ inducing the two cuts (for the definition of sequences inducing cuts, see definition 4.1). We will use this result in subsequent sections.

Definition 3.1. 1) Let \mathcal{F} be a real closed field. A cut is a pair $C = (C^-, C^+)$ such that $C^-, C^+ \subseteq \mathcal{F}$, C^- an initial segment of \mathcal{F} , C^+ an end segment of \mathcal{F} and $C^- \cup C^+ = \mathcal{F}$, $C^- \cap C^+ = \emptyset$.

2) A cut $C = (C^-, C^+)$ will be called Dedekind if C^- has no maximal element and C^+ has no minimal element.

3) We define the cofinality of the cut C to be the pair $(\text{cf}(C^-), \text{cf}(C^+, *))$ where $C^+, *$ is C^+ going backwards. $\text{cf}(C^-)$ will be called the left cofinality of C and right cofinality is defined in the same way.

Definition 3.2. Let \mathcal{F} be a real closed field and let $C = (C^-, C^+)$ be a cut in it. Let \mathcal{K} be a real closed field extending \mathcal{F} . We say that $a \in \mathcal{K}$ realizes the cut C if $c < a$ for every $c \in C^-$ and $a < d$ for every $d \in C^+$. In this case we say that \mathcal{K} has realization of C .

Definition 3.3. Let \mathcal{F} be a real closed field and let S be a family of cuts in \mathcal{F} . We say that S is dependent if for some real closed field extending \mathcal{F} with realizations $\{a_s : s \in S\}$ of the cuts in S we have that $\{a_s : s \in S\}$ is algebraically dependent over \mathcal{F} , i.e. there exists some $n < \omega$ and polynomial $p(\bar{x}) \neq 0$ in n variables with parameters from \mathcal{F} such that some sequence of length n of elements in $\{a_s : s \in S\}$ is a solution to $p(\bar{x}) = 0$. Otherwise, the set is said to be independent.

The reader can easily check that this definition is sound.

Claim 3.4. Let \mathcal{F} be a real closed field and S a set of cuts in \mathcal{F} . Then S is independent iff whenever $D \subseteq S$, $\mathcal{K} \geq \mathcal{F}$ and $\{a_d : d \in D\} \subseteq \mathcal{K}$ realize the cuts in D then the real closure of $\mathcal{F}(\{a_d : d \in D\})$ does not realize any cut from $S \setminus D$.

Proof. First assume that S is independent and let D be a subset of S . Take some \mathcal{K} a real closed field extending \mathcal{F} with realizations $\{a_d : d \in D\}$ to the cuts in D . We claim that the real closure of $\mathcal{F}(\{a_d : d \in D\})$ realizes no type in $S \setminus D$. Assume otherwise, then some cut C in $S \setminus D$ is realized. Hence some polynomial $p(x)$ with parameters from $\mathcal{F}(\{a_d : d \in D\})$ witness it. Now $p(x)$ can be rewritten as $p(x, \bar{d})$ with parameters from \mathcal{F} , where \bar{d} is a sequence from $\{a_d : d \in D\}$. So $p(\bar{x}, \bar{y})$ witness that S is algebraically dependent over \mathcal{F} . Contradiction.

For the second direction assume S is dependent. Then there exists some polynomial $p(x) \neq 0$ with parameters from \mathcal{F} such that some finite sequence from $\{a_s : s \in S\}$ solves $p(\bar{x}) = 0$, assume $p(\bar{d}) = 0$, $\bar{d} = \langle d_1, \dots, d_n \rangle$ and d_i realize the cut D_i in S for $i = 1, \dots, n$. Without loss of generality $p(x, d_2, \dots, d_n) \neq 0$ and so D_1 is realized in the real closure of $\mathcal{F}(d_2, \dots, d_n)$ and we are done. \square

Definition 3.5. Let $C = (C^-, C^+)$ be a Dedekind cut in some real closed field \mathcal{F} .

1) C is said to be positive if $C^- \cap \mathcal{F}^+ \neq \emptyset$.

2) A positive cut C is said to be additive if C^- is closed under addition.

3) C is said to be multiplicative if $C^- \cap F^+$ is closed under multiplication and $2 \in C^-$.

4) C is said to be a Scott cut if for every $t > 0$ in \mathcal{F} we have some $a \in C^-, b \in C^+$ with $b - a < t$.

Claim 3.6. *Let $\{C_1, C_2\}$ be a set of Dedekind cuts in some real closed field \mathcal{F} . So the following are equivalent:*

- (1) $\{C_1, C_2\}$ is dependent
- (2) for some polynomial $p(x, y, c)$ with $\bar{c}, d \in \mathcal{F}$ we have that $\varphi(x, y, \bar{c}, d) = “y \text{ is the smallest set to realize } y > d \text{ and } p(x, y, \bar{c}) = 0”$ defines a strictly monotonic function $y = f(x)$ from some interval I_1 around C_1 onto some interval I_2 around C_2 , such that the cut is respected, i.e. $f(I_1 \cap C_1) = I_2 \cap C_2^-$ or $f(I_1 \cap C_1^-) = I_2 \cap C_2^+$
- (3) some \mathcal{F} -definable function $\varphi(x, y, \bar{c})$ monotonically maps some interval around C_1 onto some interval around C_2 such that the cut is respected.

Proof. (1) \Rightarrow (2) First assume $\{C_1, C_2\}$ is dependent. By definition for some a, a^* realizations of C_1, C_2 respectively in some real closed field \mathcal{F}^+ extending \mathcal{F} , and for some polynomial $p(x, y_1) \neq 0$ we have that $\models p(a, a^*) = 0$. Now let $b \in \mathcal{F}^+$ be the smallest root of $p(a, y)$ in \mathcal{F}^+ that is larger than C_2^- and let $d \in C_2^-$ be some element smaller than b and larger than all the smaller roots of $p(a, y)$ in \mathcal{F}^+ . Now take $\varphi(x, y) = “y \text{ is the smallest element to realize } y > d \wedge p(x, y) = 0”$.

Now surely $\models \varphi[a, b]$ and by Corollary 1.8 we have that $\varphi(x, y)$ defines y as a function of x . By quantifier elimination the function is monotonic and the cut is respected as $\models \varphi[a, b]$.

(3) follows from (2) trivially.

(3) \Rightarrow (1) Now assume a function such as in (3) exists, and denote it $f(x) = y$. Let c_1 be some realization of C_1 in some real closed field extending \mathcal{F} . Now $\mathcal{F} \models “\varphi(x, y, \bar{c}) \text{ defines a monotonic function on } (d^-, d^+)”$ where $d \in C_1, d^+ \in C_1^+$. Hence c_1 has an image under $f(x)$, denoted $f(c_1)$. Without loss of generality the function is strictly increasing in (d^-, d^+) . So for every $a \in f(d^-, d^+) \cap C_2^-$ we have that $f(x) > a$ on some end-segment of C_1 and so $f(c_1) > a$ as the function is monotonic. However, for every $b \in f(d^-, d^+) \cap C_2$ by similar considerations we have that $f(c_1) < b$. So $f(c_1)$ realize the cut C_2 and so every field extending \mathcal{F} realizing C_1 also realize C_2 and so the cuts are dependent. \square

Definition 3.7. Two Dedekind cuts C_1, C_2 in some real closed field \mathcal{F} will be called equivalent if $\{C_1, C_2\}$ is dependent. In this case we will say that C_i is equivalent to C_{3-i} for $i = 1, 2$. They are positively equivalent if there exists some \mathcal{F} -definable order-preserving function from some interval about C_1 to some interval about C_2 respecting the cuts and negatively equivalent if there exists such an anti-order-preserving function.

Remark 3.8. Every pair of equivalent cuts is either positively equivalent, negatively equivalent or both.

Remark 3.9. Positive equivalence is a transitive relation between Dedekind cuts (the proof is immediate from the last claim).

The following claim strengthens the previous claim:

Corollary 3.10. *Let $\{C_1, C_2\}$ be a set of Dedekind cuts in some real closed field \mathcal{F} . So C_1 is positively equivalent to C_2 iff there exists some \mathcal{F} -definable function monotonically mapping some end-segment of C_1^- onto some end-segment of C_2^- .*

Proof. The first direction follows from the Claim 3.6.

For the second direction let $\varphi(x, y, \bar{c})$ be such a function and without loss of generality assume $\varphi(x, y, \bar{c})$ is strictly increasing on D^- an end segment of C_1^- . Now examine the formula $\psi(x, \bar{c}) \equiv \text{"}\varphi(x, y, \bar{c}) \text{ defines } y \text{ as a strictly increasing function of } x \text{ on some interval } x - \varepsilon, x + \varepsilon\text{"}$. By quantifier elimination $\{x \in \mathcal{F} : \models \psi(x, \bar{c})\}$ is a finite union of intervals and is non-empty on some end-segment of C^- , hence is non-empty on some (d^-, d^+) with $d^- \in C^-$, $d^+ \in C^+$. Now use the previous claim. \square

In dependent theories, the average type (see Definition 2.1) was defined for indiscernible sequences. The reason is that for general sequences there may exist a formula $\varphi(x, \bar{c})$ such that its truth value shifts back and forth and the average will not be defined, resulting in incomplete types. In the theory of real closed fields, by quantifier elimination, it is enough for the sequence to be endless strictly increasing (or decreasing) for the average type to be complete.

Definition 3.11. Let \mathcal{F} be some real closed field and let $\bar{c} = \langle c_t : t \in I \rangle$ be some endless increasing sequence in \mathcal{F} . For every $\Delta \subseteq \mathbb{L}(T)$ we define the Δ -average type of \bar{c} over $A \subset C$ to be:

$$\text{Av}_\Delta(A, \bar{c}) = \{\varphi(x, \bar{d})^t : \bar{d} \subseteq A, \varphi(x, \bar{Y}) \in \Delta, \\ \text{for every large enough } t \in I \text{ we have } \models \varphi(c_t, \bar{d})^t\}$$

omitting the Δ means for $\Delta = \mathbb{L}(T)$.

Corollary 3.12. *Let \mathcal{F} be some real closed field and let $\bar{c} = \langle c_t : t \in I \rangle$ be some endless increasing sequence in \mathcal{F} . So for every $A \subset \mathcal{C}$ we have that $\text{Av}(\bar{c}, A)$ is a complete type over A .*

Proof. Let $\varphi(x, \bar{y}) \in \mathbb{L}$ be some formula and $\bar{d} \in A$. By quantifier elimination we have that $\{x \in \mathcal{F} : \models \varphi(x, \bar{d})\}$ is a finite union of intervals in \mathcal{F} . Hence for some truth value \mathbf{t} for some end-segment J of I we have that $\forall t \in J (\models \varphi(c_t, \bar{d})^{\mathbf{t}})$. So we are done. \square

We leave the proof of the following claim to the reader.

Claim 3.13. *Let C be a Dedekind cut in some real closed field \mathcal{F} and let $\bar{c} = \langle c_t : t \in \alpha \rangle$ be some strictly increasing unbounded sequence in C^- . So $a \in \mathcal{C}$ realizes \mathcal{C} in some real closed field extending \mathcal{F} iff a realizes $\text{Av}(\mathcal{F}, \bar{c})$.*

Claim 3.14. *Let $\{C_1, C_2\}$ be a set of Dedekind cuts in some real closed field \mathcal{F} . Assume that for some polynomial $p(x, y, \bar{c}) \neq 0$ with $\bar{c} \in \mathcal{F}$ we have that there exists some unbounded increasing sequence $\bar{a} = \langle a_t : t \in I \rangle$ in C_1^- and some unbounded increasing sequence $\bar{b} = \langle b_t : t \in I \rangle$ in C_2^- such that $p(a_t, b_t, \bar{c}) = 0$ for every $t \in I$. Then C_1, C_2 are positively equivalent.*

Proof. Without loss of generality we can assume that there exists no $a \in C_1^-$ such that $\forall b (p(a, b, \bar{c}) = 0)$ (there can be only a finite number of such a 's). Let a realize C_1 in some real closed field extending \mathcal{F} . So by quantifier elimination, for some end-segment D_2 of C_2^- and $n \in \omega$ we have that $(\forall z \in D_2)(p(a, y, \bar{c}) = 0 \text{ has exactly } n$

solutions bigger than z). Now for every $b \in C_2^-$ for every large enough $s \in I$ we have $b_s > b$, so for every large enough $t \in I$ we have $\exists y(p(a_t, y, \bar{c}) = 0, y > b) \in \text{Av}(\mathcal{F}, \bar{\mathbf{a}})$. So $n > 0$. Without loss of generality for some $e \in D_2$ we have $\forall s \in I(b_s > e)$. Again without loss of generality we may assume that for every $t \in I$ there are exactly n solutions to $p(a_t, y, \bar{c}) = 0$ bigger than e and by quantifier elimination we have that for some end-segment D_1 of C_1^- we have that $\forall x \in D_1(p(x, y, \bar{c}) = 0$ has exactly n solutions bigger than e).

Let $\varphi(x, y, \bar{c}) =$ “ y is the smallest solution to $p(x, y, \bar{c}) = 0$ bigger than e ”. So $\varphi(x, y, \bar{c})$ defines y as a function of x for $x \in D_1$. Without loss of generality $\varphi(x, y, \bar{c})$ is strictly monotonic on D_1 and $\bar{\mathbf{a}} \subset D_1$. Assume by contradiction that $\varphi(x, y, \bar{c})$ is strictly decreasing and let $d \in D_1$ and $\models \varphi[d, b, \bar{c}]$. So for every $x \in D_1$ bigger than d we have that $p(x, y, \bar{c}) = 0$ has a solution in (e, b) . Hence for every large enough $t \in I$ we have that $p(a_t, y, \bar{c}) = 0$ has at most $n - 1$ solutions bigger than b . So $p(a, y, \bar{c}) = 0$ has at most $n - 1$ solutions bigger than b , contradicting the choice of D_2 . So $\varphi(x, y, \bar{c})$ is strictly increasing on D_1 .

Let C'_2 be the downward closure in \mathcal{F} of the image of D_1 under the function φ , so C'_2 is an initial segment with no endpoint. If we show that $C'_2 = C_2^-$ then we are done by Claim 3.10. We first show that $C'_2 \subseteq C_2^-$. For every $g \in D_1$ there exists some $t \in I$ with $a_t > g$. By assumption $p(a_t, b_t, \bar{c}) = 0$ so the image of a_t under $\varphi(x, y, \bar{c})$ is in C_2^- . Since φ is monotonically increasing then the image of g is in C_2^- , too. So the image of D_1 is a subset of C_2^- hence so is C'_2 . Now assume that $C'_2 \neq C_2^-$. So for some $b \in C_2^-$ we have $b > C'_2$. So for every large enough $x \in C_1^-$ we have that $p(x, y, \bar{c}) = 0$ has a solution in (e, b) and as before we get a contradiction to the choice of D_2 . \square

§ 4. PERPENDICULARITY IN REAL CLOSED FIELDS

In this section we explore the meaning of perpendicular sequences in real closed fields. We prove that every cut with large enough cofinality is induced by an indiscernible sequence. We show that independence of cuts in the sense of Definition 3.3 is equivalent to perpendicularity of their sequences (see Theorem 4.5 and Claim 4.10).

Definition 4.1. Let $\bar{a} = \langle a_t : t \in I \rangle$ be some strictly increasing indiscernible sequence. The cut induced by \bar{a} in \mathcal{F} is the cut (C^-, C^+) with C^- the downwards closure of \bar{a} in \mathcal{F} .

Claim 4.2. Let (C^-, C^+) be a Dedekind cut in some real closed field \mathcal{F} with left cofinality $\lambda > \aleph_0$. Then there exists some indiscernible sequence $\bar{a} \subset \mathcal{F}$ with $\text{cf}(\bar{a}) = \lambda$ such that C is induced by \bar{a} .

Proof. Let $\langle d_t : t \in \lambda \rangle$ be some cofinal sequence in C^- . Let D be some ultrafilter on C^- extending the family of end sections of C^- . We now find an endless indiscernible sequence $\bar{b} \subseteq \mathcal{C}$ of order type ω based on D over \mathcal{F} . We now define by induction $a_t \in C^-$ for $t \in \lambda$ such that for every $t \in \lambda$, a_t realizes $\text{Av}(\bar{b} \cup \{a_s : s < t\}, \bar{b})$ and $d_t < a_t$. If we succeed then $\bar{a} = \langle a_t : t \in \lambda \rangle$ is as required (it is indiscernible as a sequence based on \bar{b}). At the t -th stage, for every $\varphi(x) \in \text{Av}(\bar{b} \cup \{a_s : s < t\}, \bar{b})$ there exists by definition of average some b_s with $\models \varphi(b_s)$ and s is bigger than all the indices of the b_s 's appearing as parameters in $\varphi(x)$. Recall that b_s realizes $\text{Av}(C^- \cup \{b_\ell : \ell < s\}, D)$ and by the choice of D there exists some unbounded $A_\varphi \subset C^-$ with each $a \in A_\varphi$ realizing $\varphi(x)$, $a > d_t$.

By quantifier elimination of the theory of real closed fields, for every model \mathcal{T} and formula $\chi(x)$ with parameters from \mathcal{T} we have that $\{\chi(x) : x \in \mathcal{T}\}$ is a finite union of convex subsets of \mathcal{T} . So without loss of generality we choose A_φ such that A_φ is an end-section of C^- . Now $|\{\bar{b} \cup \{a_s : s < t\}| = \aleph_0 < \text{cf}(C^-)$ so there exists some $A_t \subset C^-$ an end-section with each $a \in A_t$ realizing $\text{Av}(\bar{b} \cup \{a_s : s < t\}, \bar{b})$. Take some $a_t \in A_t$. So the induction is successful and we are done. \square

Question 4.3. Is this true without assuming $\lambda > \aleph_0$?

Answer 4.4. No. Consider some \aleph_1 -saturated real-closed field \mathcal{S} . Take C^- the downward closure of $\mathbb{N} \subset \mathcal{S}$. Surely it's a Dedekind cut, and no indiscernible sequence induce it. However, if we restrict to finite $\Delta \subseteq \mathbb{L}(T)$ there exists a Δ -indiscernible sequence inducing the cut. See the original proof.

Theorem 4.5. Let \mathcal{F} be a real closed field, C_1, C_2 Dedekind cuts in \mathcal{F} induced by the indiscernible sequences $\bar{a} = \langle a_t : t \in I \rangle$ and $\bar{b} = \langle b_s : s \in J \rangle$ respectively. Then C_2 is positively equivalent to C_1 iff \bar{a}, \bar{b} are not \mathcal{F} -perpendicular in the sense of Definition 2.16.

Proof. First assume that C_1, C_2 are positively dependent. So there exist intervals I_1 around C_1 and I_2 around C_2 and a polynomial $p(x, y, \bar{c})$ with $\bar{c} \subset \mathcal{F}$ such that $p(x, y, \bar{c}) = 0$ defines a function from I_1 onto I_2 respecting the cuts, denoted $y = f(x)$. Define $\varphi(x, y, \bar{c}) = "f(x) = z \text{ and } y \geq z"$. Now for every large enough $t \in I$ we have $f(a_t) \in C_2$. Hence for some large enough $s \in J$ we have $b_s > f(a_t)$ and so $\models \varphi(a_t, b_s, \bar{c})$. On the other hand for every large enough $s \in J$ for every large enough $t \in I$ we have $f(a_t) > b_s$ and so $\models \neg \varphi(a_t, b_s, \bar{c})$. So by definition \bar{a}, \bar{b} are not \mathcal{F} -perpendicular.

For the second direction, we will use Claim 3.14. So $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are not \mathcal{F} -perpendicular. By quantifier elimination for some polynomial $p(x, y, \bar{c})$ we have that

- (1) for every large enough $t \in I$ for every large enough $s \in J$ we have $p(a_t, b_s, \bar{c}) > 0$
- (2) for every large enough $s \in J$ for every large enough $t \in I$ we have $p(a_t, b_s, \bar{c}) > 0$.

We shall now find an ordinal λ and $\bar{\mathbf{a}} = \langle t_\alpha : \alpha < \lambda \rangle, \bar{\mathbf{b}}' = \langle s_\alpha : \alpha < \lambda \rangle$ unbounded sequences of I with the property that for every $\alpha, \beta \in \lambda$ we have $\models p(a_{t_\alpha}, b_{s_\beta}, \bar{c}) > 0$ iff $\beta \geq \alpha$. So let $\lambda = \text{cf}(I)$ and $\bar{\mathbf{c}} = \langle c_\alpha : \alpha < \lambda \rangle$ be an unbounded sequence in I . We will define $\langle t_\alpha : \alpha < \lambda \rangle, \langle s_\alpha : \alpha < \lambda \rangle$ by induction on $\alpha < \lambda$ such that $t_\alpha > c_\alpha$ for every $\alpha < \lambda$. For $\alpha = 0$ let t_0 be large enough as in (1) and larger than c_0 . Let s_0 be large enough as in (1) for t_0 and large enough as in (2). For limit ordinals $\alpha < \lambda$ we know that for every $\beta < \alpha$ for b_{s_β} we have some large enough t_{s_β} such as in (2) and since $a < \text{cf}(I)$ some t_α is large enough as in (2) for every $\beta < \alpha$ and larger than $\langle t_\beta : \beta < \alpha \rangle$ and larger than c_α . Let s_α be large enough as in (1) for t_α . For $\alpha = \beta + 1$ let t_α be large enough as in (2) for b_{s_β} and larger than c_α . Let s_α be large enough as in (1) for t_α . One can easily check that the condition that for every $\alpha, \beta \in \lambda$ we have $\models p(a_{t_\alpha}, b_{s_\beta}, \bar{c}) > 0$ iff $\beta \geq \alpha$, $\bar{\mathbf{a}}'$ is unbounded since $t_\alpha > c_\alpha$ for every $\alpha < \lambda$ and $\bar{\mathbf{c}} = \langle c_\alpha : \alpha < \lambda \rangle$ is unbounded in I . We will now prove that $\bar{\mathbf{b}}'$ is unbounded in J . Assume otherwise, then for some large enough $s \in UJ$ we have that s is large enough as in (1) for every $\langle t_\alpha : \alpha < \lambda \rangle$ and s is larger than $\langle s_\alpha : \alpha < \lambda \rangle$.

Now take $t \in I$ large enough as in (2) for s , so $p(a_t, b_s, \bar{c}) < 0$. We know that $t < t_\alpha$ for some $\alpha < \lambda$ since $\bar{\mathbf{a}}'$ is unbounded, so $p(a_{t_\alpha}, b_s, \bar{c}) < 0$. Now s_α is large enough as in (1) for t_α and $s > s_\alpha$ so $p(a_{t_\alpha}, b_s, \bar{c}) > 0$ -contradiction.

Now for every $\alpha < \lambda$ we have that $p(a_{t_\alpha}, b_{s_\alpha}, \bar{c}) > 0$ and $p(a_{t_{\alpha+1}}, b_{s_\alpha}, \bar{c}) < 0$ so for some $d_\alpha \in (a_{t_\alpha}, a_{t_{\alpha+1}})$ we have $p(d_\alpha, b_{s_\alpha}, \bar{c}) = 0$. Now by Claim 3.14 we are done. \square

Claim 4.6. *Let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle$ be an endless strictly increasing sequence in some real closed field \mathcal{F} . Then for some \mathcal{S} a real closed field extending \mathcal{F} we have that $\bar{\mathbf{a}}$ induces a Dedekind cut in \mathcal{S} and in every real closed field extending \mathcal{S} .*

Proof. Take \mathcal{S} to be a real closed field extending \mathcal{F} with a realization of the type $\{x > 0\} \cup \{x < a_t - a_s : t, s \in I, t > s\}$ and of the type $\{x > a_t : t \in I\}$. The rest is left to the reader. \square

Corollary 4.7. *In the theory of real closed fields, not being perpendicular is a transitive relation on endless indiscernible sequences of elements.*

Proof. Let $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}$ be endless indiscernible sequences of elements in some real closed field \mathcal{F} such that $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are not perpendicular and $\bar{\mathbf{b}}, \bar{\mathbf{c}}$ are not perpendicular. So for some real closed \mathcal{S} we have that $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are not \mathcal{S} -perpendicular, $\bar{\mathbf{b}}, \bar{\mathbf{c}}$ are not \mathcal{S} -perpendicular and that $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}$ induce Dedekind cuts in \mathcal{S} , denoted $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$. So by the previous theorem $\mathcal{C}_a, \mathcal{C}_b$ are positively equivalent, and $\mathcal{C}_b, \mathcal{C}_c$ are positively equivalent and by transitivity $\mathcal{C}_a, \mathcal{C}_c$ are positively equivalent. So $\bar{\mathbf{a}}, \bar{\mathbf{c}}$ are not \mathcal{S} -perpendicular and so are not perpendicular. \square

Example 4.8. Take $T = \text{Th}(\mathbb{R}^2, <_1, <_2)$ (so T is dependent because it can be interpreted in \mathbb{R} and \mathbb{R} is dependent) and

$$\bar{\mathbf{a}} = \langle (n, 0) : n \in \mathbb{N} \rangle, \bar{\mathbf{b}} = \langle (n, n) : n \in \mathbb{N} \rangle, \bar{\mathbf{c}} = \langle (0, n) : n \in \mathbb{N} \rangle.$$

Remark 4.9. In the next Claim we replace \mathcal{F} -perpendicular Definition 2.16, in Theorem 4.5 by preperpendicular, Definition 2.8 is a special case.

Claim 4.10. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two Dedekind cuts in some real closed field \mathcal{F} such that the left cofinality of \mathcal{C}_ℓ is strictly smaller than the right cofinality of \mathcal{C}_ℓ and larger than \aleph_0 for $\ell \in \{1, 2\}$. Let $\bar{\mathbf{a}}^1 = \langle a_t^1 : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle a_t^2 : t \in I^2 \rangle$ induce $\mathcal{C}_1, \mathcal{C}_2$ respectively. So \mathcal{C}_1 is positively equivalent to \mathcal{C}_2 iff $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are not perpendicular.*

Proof. First assume that \mathcal{C}_1 is positively equivalent to \mathcal{C}_2 . So by Theorem 4.5 we have that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are not \mathcal{F} -perpendicular and so are not perpendicular.

Now assume $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are not perpendicular and for a contradiction assume \mathcal{C}_1 is not positively equivalent to \mathcal{C}_2 . First, by Theorem 2.19 we have that $\text{cf}(\bar{\mathbf{a}}^1) = \text{cf}(\bar{\mathbf{a}}^2)$ and by Theorem 4.5 we conclude that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are \mathcal{F} -perpendicular. We will use Claim 2.18 to show that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular. We will choose by induction on $n \in \omega$ two elements b_n^1, b_n^2 of \mathcal{F} such that the conditions in the claim occur.

So in the n -th stage, we first find $b_n^1 \in \mathcal{F}$ which realize

$$q(x) = \text{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{b_m^k : k \in \{1, 2\}, m < n\}, \bar{\mathbf{a}}^\ell).$$

So for every $\varphi(x) \in q(x)$ for every large enough $t \in I^1$ we have $\models \varphi[a_t]$. By quantifier elimination the set $\{x \in \mathcal{F} : \models \varphi[x]\}$ is a finite union of intervals. So for some $b_\varphi \in \mathcal{C}_1^+$ we have that for every $c \in \mathcal{C}_1^+$ such that $c < b_\varphi$ we have $\models \varphi[c]$. Since $|q(x)| = \text{cf}(\bar{\mathbf{a}}^1)$ and the latter is smaller than the right cofinality of \mathcal{C}_1 then for some $b \in \mathcal{C}_1^+$ we have that $b < b_\varphi$ for every $\varphi \in q$. In other words b realizes q . Now just denote $b_n^1 = b$ and we are done. b_n^2 is chosen in the same manner.

This contradiction concludes the proof. \square

§ 5. STRONG PERPENDICULARITY IN REAL CLOSED FIELDS

In this item we explore strong perpendicularity of sequences in real closed fields. We first define neighbor sequences and strong perpendicularity. We then show that strong perpendicularity is invariant to reversing the order of a sequence, applying a definable function and to taking pre-images of definable functions. The section ends with the result that there are no strongly-perpendicular sequences in real closed fields (Theorem 5.17).

Definition 5.1. 1) Two indiscernible sequences $\langle a_t^\ell : t \in I^\ell, \ell \in \{1, 2\} \rangle$ will be called immediate neighbors (or inb's for short) if there exists some indiscernible sequence $\langle b_s : s \in J \rangle$ and order preserving or anti-order preserving injections $\sigma^\ell : I^\ell \rightarrow J$ such that for $\ell \in \{1, 2\}, t \in I^\ell$ we have $a_t^\ell = b_{\sigma^\ell(t)}$. In other words, two sequences are immediate neighbors if they can be embedded in the same indiscernible sequence.

2) Two indiscernible sequences $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ will be called neighbors (or nb's for short) if there is a finite sequence $\bar{\mathbf{b}}^0, \bar{\mathbf{b}}^1, \dots, \bar{\mathbf{b}}^\ell$ of indiscernible sequences such that $\bar{\mathbf{a}}^1 = \bar{\mathbf{b}}^0, \bar{\mathbf{a}}^2 = \bar{\mathbf{b}}^\ell$ and $\bar{\mathbf{b}}^i, \bar{\mathbf{b}}^{i+1}$ are inb's for $i = 0, \dots, \ell - 1$. In this case we say that the sequences are ℓ -nb's. Note that to say that two sequences are 1-nb's is the same as saying that they are inb's.

Recall Definition 3.5.

Example 5.2. Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ be two indiscernible sequences in some real closed field \mathcal{F} such that both induce multiplicative cuts. Then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are 2-nb's.

Proof. The proof is based on the claim below.

We construct a third sequence $\bar{\mathbf{b}} = \langle b_s : s \in \omega \rangle$ such that $\bar{\mathbf{a}}^1, \bar{\mathbf{b}}, \bar{\mathbf{a}}^2, \bar{\mathbf{b}}$ are both indiscernible. This will prove the claim with the sequence $\bar{\mathbf{a}}^1, \bar{\mathbf{b}}, \bar{\mathbf{a}}^2$ witnessing that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are indeed 2-nb's. So we begin with some endless well-ordered set ω and construct b_s by induction on $s \in \omega$. Assume b_t has already been defined for every $t < s$. We define b_s to be some element from \mathcal{C} realizing the type

$$\{x > p(a_{q_0}^1, \dots, a_{q_m}^1, a_{r_0}^2, \dots, a_{r_n}^2, b_{t_0}, \dots, b_{t_k}) : \begin{array}{l} p \text{ is some polynomial,} \\ a_{q_0}^1, \dots, a_{q_m}^1 \text{ is a subsequence of } \bar{\mathbf{a}}^1, \\ a_{r_0}^2, \dots, a_{r_n}^2 \text{ is a subsequence of } \bar{\mathbf{a}}^2, \\ t_0 <_J \dots <_J t_k <_J s \end{array}\}.$$

Now use the claim below to show that $\bar{\mathbf{a}}^1, \bar{\mathbf{b}}, \bar{\mathbf{a}}^2, \bar{\mathbf{b}}$ are both indiscernible. □

Claim 5.3. Let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle$ be some increasing sequence of positive elements in some real closed field \mathcal{F} . Then $\bar{\mathbf{a}}$ is indiscernible and induces a multiplicative cut in \mathcal{F} iff for every $k \in \mathbb{N}, t_0 <_I \dots <_I t_{k-1} <_I t_k$ we have that $\models \{a_{t_k} >_{\mathcal{F}} P(a_{t_0}, \dots, a_{t_{k-1}}) : P(x_0, \dots, x_{k-1}) \text{ is some parameter-free polynomial}\}$.

Proof. First assume that $\bar{\mathbf{a}}$ is indiscernible and induce a multiplicative cut in \mathcal{F} . Let $k \in \mathbb{N}, P(x_0, \dots, x_{k-1})$ some polynomial and $t_0 <_I \dots <_I t_k$ induces from I and we will prove that $\models a_{t_k} >_{\mathcal{F}} P(a_{t_0}, \dots, a_{t_{k-1}})$. We know that $\bar{\mathbf{a}}$ is indiscernible so it is enough to prove the above for some increasing sub-sequence in I of length $k + 1$. Now $P(x_0, \dots, x_{k-1}) = \sum_{i=0}^m q_i(x_0, \dots, x_{k-1})$ where q_i are monomial. $\bar{\mathbf{a}}$ induces a multiplicative cut so for every $i \in \{0, \dots, m\}$ there exists some $s_i \in$

$I, s_i >_I t_{k-1}$ such that $a_{s_i} > q_i(a_{t_0}, \dots, a_{t_{k-1}})$. Since $\bar{\mathbf{a}}$ also induces an additive cut (the reader can easily check that every multiplicative cut is also an additive cut) for some $s \in I, s >_I t_{k-1}$ we have that $a_s > \sum_{i=0}^m a_{s_i}$. All together we have $a_s > \sum_{i=0}^m q_i(a_{t_0}, \dots, a_{t_{k-1}}) = P(a_{t_0}, \dots, a_{t_{k-1}})$ and we are done.

For the second direction we first prove that $\bar{\mathbf{a}}$ is indiscernible. Denote by \mathbb{P} the set of all polynomials. By assumption we have that $\bar{\mathbf{a}}$ is (\mathbb{P}, ϕ) -indiscernible. By quantifier elimination for the theory of real closed fields we have that $\bar{\mathbf{a}}$ is indiscernible. To prove that $\bar{\mathbf{a}}$ induces a multiplicative cut we denote by (C^-, C^+) the cut induced by $\bar{\mathbf{a}}$ and assume $x, y \in \mathcal{F}$ such that $x, y \in C^- \cap \mathcal{F}^+$. By definition we have that for some $t \in I : a_t > x, y$. Let $s >_I t$. So by assumption $a_s > p(a_t)$ for every polynomial p , specifically $p(x) = x^2$. So $a_s > a_t^2 > x \cdot y$ (remember that all the elements discussed here are positive) and specifically $x \cdot y \in C^-$. Hence (C^-, C^+) is multiplicative and we are done. \square

Claim 5.4. *Let $\langle a_t : t \in I \rangle$ be some indiscernible sequence, and denote by I^* the set I ordered in reversed order. So $\langle a_t : t \in I \rangle, \langle a_t : t \in I^* \rangle$ are immediate nb's.*

Proof. The identity function from I to itself is enough - just look at the definitions. \square

Notation 5.5. Let $\bar{\mathbf{a}}$ be some infinite indiscernible sequence. We denote by $\text{tp}'(\bar{\mathbf{a}})$ the type of some infinite countable subsequence of $\bar{\mathbf{a}}$, we call $\text{tp}'(\bar{\mathbf{a}})$ the local type of $\bar{\mathbf{a}}$.

Claim 5.6. *Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ be some ℓ -nb's indiscernible sequences for some $\ell \in \mathbb{N}$ then $\text{tp}'(\bar{\mathbf{a}}^1) = \text{tp}'(\bar{\mathbf{a}}^2)$ or $\text{tp}'(\bar{\mathbf{a}}^1) = \text{tp}'(\bar{\mathbf{a}}^{2,*})$. In other words, the local type is preserved or reversed between neighbors.*

Proof. Using induction on ℓ we are left with proving that the claim is correct in the case of immediate neighbors. This case is also easy since if $\bar{\mathbf{a}}^i$ is embedded in $\bar{\mathbf{b}}$, then surely $\text{tp}'(\bar{\mathbf{b}}) = \text{tp}'(\bar{\mathbf{a}}^i)$. \square

We next turn our attention to a different view of perpendicularity; one which eliminates, in some sense, the dependency of the definition of perpendicularity in the particular choice of the sequences themselves.

Definition 5.7. Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ be some endless indiscernible sequences in some real closed field \mathcal{F} . We say that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are strongly perpendicular iff $\bar{\mathbf{c}}^1, \bar{\mathbf{c}}^2$ are perpendicular whenever $\bar{\mathbf{a}}^i, \bar{\mathbf{c}}^i$ are nb's for $i \in \{1, 2\}$.

Example 5.8. Consider the following model $M : |M| = \{(x, y) : x, y \in \mathbb{Q}\}$, $<_1$ is a binary predicate agreeing in every point with the natural order of \mathbb{Q} according to the first element of the tuple, and $<_2$ is a binary predicate agreeing in every point with the natural order of \mathbb{Q} according to the second element of the tuple.

(The reader should check that this model is indeed a model of a dependent theory). Indiscernible sequences $\langle (a_t, b_t) : t \in I \rangle$ in this model divide into 8 kinds, according to the direction in each dimension (their local types):

- (1) $\forall t <_I s (a_t = a_s, b_t <_{\mathbb{Q}} b_s)$
- (2) $\forall t <_I s (a_t = a_s, b_t >_{\mathbb{Q}} b_s)$

- (3) $\forall t <_I s(a_t >_{\mathbb{Q}} a_s, b_t = b_s)$
- (4) $\forall t <_I s(a_t <_{\mathbb{Q}} a_s, b_t = b_s)$
- (5) $\forall t <_I s(a_t <_{\mathbb{Q}} a_s, b_t <_{\mathbb{Q}} b_s)$
- (6) $\forall t <_I s(a_t <_{\mathbb{Q}} a_s, b_t >_{\mathbb{Q}} b_s)$
- (7) $\forall t <_I s(a_t >_{\mathbb{Q}} a_s, b_t <_{\mathbb{Q}} b_s)$
- (8) $\forall t <_I s(a_t >_{\mathbb{Q}} a_s, b_t >_{\mathbb{Q}} b_s)$.

In this example we will show that two indiscernible sequences of the 5th kind are always 2-nb's, that indiscernible sequences of the 1st kind are strongly perpendicular to indiscernible sequences of the 3rd kind and not strongly perpendicular to sequences of the 5th kind.

Start with two 5th kind indiscernible sequences: $\bar{\mathbf{h}}^1 = \langle h_n^1 : n \in \mathbb{N} \rangle, \bar{\mathbf{h}}^2 = \langle h_n^2 : n \in \mathbb{N} \rangle$. We construct a third sequence $\bar{\mathbf{h}} = \langle h_n : n \in \mathbb{N} \rangle$ such that $\bar{\mathbf{h}}^1, \bar{\mathbf{h}}, \bar{\mathbf{h}}^2$ are both indiscernible. This will show that indeed the sequences are 2-nb's. The construction is by induction on $n \in \mathbb{N}$. Assume h_k has been chosen for all $k < n$. Choose h_n such that $h_n >_j h_m^i$ for all $j \in \{1, 2\}, i \in \{1, 2\}, m \in \mathbb{N}$ and $h_n >_j h_k$ for all $j \in \{1, 2\}, k < n$. The reader should now check that indeed what was expected of $\bar{\mathbf{h}}$ occurs.

Now we prove that indiscernible sequences of the 1st kind are always strongly perpendicular to indiscernible sequences of the 3rd kind. We notice that since local types are preserved between neighbors and the division into kinds was based on the local type, it is enough to prove that every sequence of the 1st kind is perpendicular to every sequence of the 3rd kind. We use the fact that $\{x >_1 y, x >_2 y\}$ is an elimination set for this theory and only use this set when proving perpendicularity. So let $\bar{\mathbf{a}}^1 = \langle (x_t^1, y_t^1) : t \in I^1 \rangle, \bar{\mathbf{a}}^2 = \langle (x_s^2, y_s^2) : s \in I^2 \rangle$ be two indiscernible sequences of the 1st and 3rd kind respectively. So x_t^1 are constant and equal to some x_0^1 and y_s^2 are also constants and equal to some y_0^2 .

Now let $\varphi_i(x, y) = x >_i y$ for $i \in \{1, 2\}$. Assume that for every large enough $t \in I^1$ we have that for every large enough $s \in I^2$ the following holds: $(x_t^1, y_t^1) >_1 (x_s^2, y_s^2)$. So for every large enough $t \in I^1$ for every $s \in I^2$ we have that $x_0^1 >_{\mathbb{Q}} x_s^2$ in particular for every large enough $s \in I^2$ for every large enough $t \in I^1 : (x_t^1, y_t^1) >_1 (x_s^2, y_s^2)$. The opposite case as well as the cases involving $>_2$ are handled the same way, and since this is an elimination set, we are done.

Here we shall prove that sequences of the 1st kind are not strongly perpendicular to sequences of the 5th kind. Let $\bar{\mathbf{a}}^1 = \langle (x_0^1, y_t^1) : t \in I \rangle$ be a sequence of the 1st kind. By definition in order to prove our claim it is enough to show that $\bar{\mathbf{a}}^1$ is not perpendicular to some sequence of the 5th kind which is a neighbor of a given sequence. However, we showed earlier in this example that any two sequences of the 5th kind are nb's, so it is enough to show that $\bar{\mathbf{a}}^1$ is not perpendicular to some sequence $\bar{\mathbf{a}}^2$ of the 5th kind. So let $\langle x_t^2 : t \in I \rangle$ be some increasing sequence in \mathbb{Q} . We choose $\bar{\mathbf{a}}^2 = \langle (x_t^2, y_t^1) : t \in I \rangle$. The formula $\varphi(x, y) = x >_2 y$ now witness that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are not perpendicular and we are done.

Definition 5.9. Let I be some well-ordered set and $k \in \omega$. We say that $I' \subseteq I$ is k -spaced in I if I' is well-ordered and $t +_I k <_I t'$ whenever $t < t'$ are both in I' .

Example 5.10. $5 - \mathbb{Z}$ is 4-spaced in \mathbb{Z} .

Claim 5.11. *Let $\bar{a} = \langle a_t : t \in I \rangle$ (I well ordered) be some endless indiscernible sequence over A in some monster model \mathcal{C} and let $\varphi(x_0, \dots, x_{n-1}, y)$ be some formula with parameters from A which defines a function $f(\bar{x}) = f_\varphi(\bar{x})$ from $(\cup \bar{a})^n$ to \mathcal{C} . So any sequence of the form $\langle f(a_t, a_{t+1}, \dots, a_{t+n-1}) : t \in I' \rangle$ where $I' \subseteq I$ is n -spaced is also indiscernible over A . (Put otherwise, if we are given an n -ary definable function φ and a partition of I into a sequence J of consecutive n -tuples then the image of J under φ remains indiscernible.)*

Proof. Denote $\bar{b}_t = \langle a_t, \dots, a_{t+n-1} \rangle$ so $f(\bar{a}) = \langle f(\bar{b}_t) : t \in I' \rangle$. Assume otherwise, so for some $\varphi(x_0, \dots, x_{n-1})$ with parameters from A we have that for some $t_0 < \dots < t_{n-1}$ and $s_0 < \dots < s_{n-1}$ we have that

$$\varphi[f(\bar{b}_{t_0}), \dots, f(\bar{b}_{t_{n-1}})] \Leftrightarrow \neg \varphi[f(\bar{b}_{s_0}), \dots, f(\bar{b}_{s_{n-1}})].$$

But the formula $f(\varphi)(x_0, \dots, x_{n-1}) = \varphi(f(x_0), \dots, f(x_{n-1}))$ is a formula over A so

$$f(\varphi)[\bar{b}_{t_0}, \dots, \bar{b}_{t_{n-1}}] \Leftrightarrow f(\varphi)[\bar{b}_{s_0}, \dots, \bar{b}_{s_{n-1}}].$$

And we have a contradiction. \square

Notation 5.12. We denote $f_{I'}(\bar{a}) = \langle f(a_t, \dots, a_{t+n-1}) : t \in I' \rangle$.

Claim 5.13. *Let $\bar{a} = \langle a_t : t \in I \rangle$ (I is well ordered) be some infinite indiscernible sequence over A in some monster model \mathcal{C} and let $\varphi(\bar{x}, y)$ be some formula with parameters from A which defines a non-constant function $f(\bar{x}) = f_\varphi(\bar{x})$ on $(\cup \bar{a})^n$. 1) Let \bar{c} be some indiscernible sequence which is a nb's of some sequence $f_{I'}(\bar{a})$ where $I' \subseteq I$ is n -spaced. Then for some indiscernible sequence $\bar{a}' = \langle a'_t : t \in J \rangle$, which is a nb's of \bar{a} , we have that $\bar{c} = f_{J'}(\bar{a}')$ for some unbounded n -spaced $J' \subseteq J$. 2) Assume that \bar{a} is endless and for some indiscernible endless sequence \bar{c} we have that \bar{c} is not perpendicular to some $f_{I'}(\bar{a})$ where $I' \subseteq I$ is n -spaced and unbounded. Then \bar{a} is not perpendicular to \bar{c} .*

Proof. 1) It is enough to prove the claim for the case when \bar{c} and $f_{I'}(\bar{a})$ are inb's. The general case is easily concluded by induction on the length of the sequence of inb's between \bar{c} and $f(\bar{a})$. So let \bar{b} be an indiscernible sequence such that σ is an order-preserving injection of \bar{c} into \bar{b} and τ is an order-preserving injection of $f_{I'}(\bar{a})$ in \bar{b} (the case where one or both injections are anti-order-preserving is proven in the same manner). We will find an indiscernible sequence \bar{a}_* with indices set I_* such that $\bar{a} \subseteq \bar{a}_*$ and $f_{I'}(\bar{a}_*) = \bar{b}$ for some unbounded n -spaced $I'_* \subseteq I_*$. This is easily done using compactness and the fact that \bar{a}, \bar{b} are indiscernible and infinite. Now all we have to do is take the subsequence of \bar{a}_* corresponding to the image of \bar{c} in \bar{b} under τ .

2) $\bar{c} = \langle c_s : s \in J \rangle$ is not perpendicular to $f_{I'}(\bar{a}) = \langle f(a_t, \dots, a_{t+n-1}) : t \in I' \rangle$, hence some $\psi(x, y)$ witness it, i.e. for every large enough $t \in I'$ for every large enough $s \in J$ we have that $\models \psi[c_s, f(a_t, \dots, a_{t+n-1})]$ and for every large enough $s \in J$ for every large enough $t \in I'$ we have that $\models \neg \psi[c_s, f(a_t, \dots, a_{t+n-1})]$. Now the formula $\psi_f(x, \bar{y}) = \psi(x, f(\bar{y}))$ witness the fact that \bar{c}, \bar{a} are not perpendicular in the same manner. \square

Recall the convention on $*$ in Claim 5.4.

Claim 5.14. *Let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle$ be some positive indiscernible sequence endless in both ways inducing an additive cut in some real closed field \mathcal{F} . Then for some $\varphi(x, y)$ which defines a function f on $(\cup \bar{\mathbf{a}})^k$ we have that either $f_{I'}(\bar{\mathbf{a}})$ or $f_{I'}(\bar{\mathbf{a}}^*)$ induces a multiplicative cut in \mathcal{F} for some unbounded k -spaced $I' \subseteq I$.*

Proof. Assume for a contradiction that for every $t_3 >_I t_2 >_I t_1 >_I t_0$ we have that:

$$\frac{a_{t_1}}{a_{t_0}} = \frac{a_{t_3}}{a_{t_2}}$$

Then by indiscernibility we have that the sequence is constant, contradiction. Without loss of generality we can assume that $\bar{\mathbf{a}}$ is increasing, otherwise use $\bar{\mathbf{a}}^*$ instead. At first, we assume the following:

$$\frac{a_{t_1}}{a_{t_0}} < \frac{a_{t_3}}{a_{t_2}} \text{ whenever } t_3 >_I t_2 >_I t_1 >_I t_0$$

Now let I' be 2-spaced in I and consider $f(x, y) = \frac{y}{x}$. We want to prove that $f_{I'}(\bar{\mathbf{a}})$ induce a multiplicative cut, i.e. that it is closed under multiplication and contains 2. So let $\frac{a_{t+1}}{a_t}, \frac{a_{s+1}}{a_s}$ be two elements of $f_{I'}(\bar{\mathbf{a}})$ with $s >_I t$. (The indices $s+1, t+1$ are not well-defined and we use them here for convenience. Formally we mean that the index in the proof is in the interval $(s, s+1)$ or $(t, t+1)$). By equivalence (1) and the fact that $\bar{\mathbf{a}}$ is indiscernible we have that for some $\ell >_I s$ we have that $\frac{a_{\ell+1}}{a_\ell} > \frac{a_{s+1}}{a_s} > \frac{a_{s+1}a_{t+1}}{a_s a_t}$. Now since $\bar{\mathbf{a}}$ is additive and strictly increasing we have that $\frac{a_{\ell+1}}{a_\ell} > 2$ always. If the reverse inequality in (1) holds, we consider $f(x, y) = \frac{x}{y}$ instead and in $\bar{\mathbf{a}}^*$. Let I' be 2-spaced in I^* . Let $\frac{a_{t+1}}{a_t}, \frac{a_{s+1}}{a_s}$ be two elements in $f_{I'}(\bar{\mathbf{a}})$ with $s >_I t$ (remember that $s <_{I'} t$). By the inverse of (1) there exists $\ell >_{I^*} t$ such that $\frac{a_{\ell+1}}{a_\ell} > \frac{a_{s+1}}{a_t} > \frac{a_{s+1}a_{t+1}}{a_s a_t}$. Again $\bar{\mathbf{a}}$ induces an additive cut and is indiscernible, so $\frac{a_{t+1}}{a_t} > 2$ and we are done. \square

Claim 5.15. *Let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle$ be some positive indiscernible sequence endless in both ways in some real closed field \mathcal{F} . So for some $\varphi(x, y)$ which defines a function f on $(\cup \bar{\mathbf{a}})^k$ we have that either $f_{I'}(\bar{\mathbf{a}})$ or $f_{I'}(\bar{\mathbf{a}}^*)$ induce an additive cut in \mathcal{F} for some unbounded k -spaced $I' \subseteq I$.*

Proof. The proof is very similar to the proof of Claim 5.14. The rest of the proof is the same as the proof there. \square

From the last two claims we can conclude:

Conclusion 5.16. *Let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle$ be some positive indiscernible sequence endless in both ways. So for some $\varphi(x, y)$ which defines a function f on $(\cup \bar{\mathbf{a}})^k$ we have that either $f_{I'}(\bar{\mathbf{a}})$ or $f_{I'}(\bar{\mathbf{a}}^*)$ induce a multiplicative cut in \mathcal{F} for some unbounded k -spaced $I' \subseteq I$.*

We now turn to the main theorem in this section.

Theorem 5.17. *Let \mathcal{F} be a real closed field. Then no two indiscernible sequences in \mathcal{F} are strongly perpendicular.*

Proof. Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ be two endless indiscernible sequences in \mathcal{F} . We will show that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are not strongly perpendicular. By Claim 5.13, Clause 2 we can assume without loss of generality that both sequences are positive (otherwise consider the function $f(x) = -x$). We may also assume without loss of generality that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are

endless in both ways and increasing (since every sequence is nb's of any sequence extending it, and of its reverse). Now using Claim 5.16 we apply a definable function f on the sequences such that the sequences $f(\bar{\mathbf{a}}^1), f(\bar{\mathbf{a}}^2)$ induce multiplicative cuts. By the example above (example 5.2) $f(\bar{\mathbf{a}}^1), f(\bar{\mathbf{a}}^2)$ has nb's $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ respectively such that $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are not perpendicular. By Claim 5.13, clause 1 above we have that for some $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ nb's of $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ respectively we have that $\bar{\mathbf{b}}^i = f(\bar{\mathbf{a}}^{i'})$. By Claim 5.13, clause 2 above we have that $\bar{\mathbf{a}}^{1'}, \bar{\mathbf{b}}^2$ are not perpendicular and then by the same claim $\bar{\mathbf{a}}^{1'}, \bar{\mathbf{a}}^{2'}$ are not perpendicular. This completes the proof. \square

REFERENCES

- [DMP06] Margit Messmer David Marker and Anand Pillay, *Model theory of fields. Second edition*, Lecture Notes in Logic, Association of Symbolic Logic, La Jolla, Ca., vol. 5, A K Peters, Ltd., Wellesley, MA, 2006.
- [Hod97] Wilfrid Hodges, *A shorter model theory*, Cambridge University Press, Cambridge, 1997.
- [Sh:c] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.
- [Sh:715] ———, *Classification theory for elementary classes with the dependence property - a modest beginning*, *Scientiae Mathematicae Japonicae* **59**, No. 2; (special issue: e9, 503–544) (2004), 265–316, math.LO/0009056.
- [Sh:783] ———, *Dependent first order theories, continued*, *Israel Journal of Mathematics* **173** (2009), 1–60, math.LO/0406440.

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